

A solution to a conjecture on the rainbow connection number*

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Abstract

For a graph G , Chartrand et al. defined the rainbow connection number $rc(G)$ and the strong rainbow connection number $src(G)$ in “G. Charand, G.L. John, K.A. Mckeon, P. Zhang, Rainbow connection in graphs, *Mathematica Bohemica*, 133(1)(2008) 85-98”. They raised the following conjecture: for two given positive integers a and b , there exists a connected graph G such that $rc(G) = a$ and $src(G) = b$ if and only if $a = b \in \{1, 2\}$ or $3 \leq a \leq b$. In this short note, we will show that the conjecture is true.

Keywords: edge-colored graph, (strong) rainbow coloring, (strong) rainbow connection number.

AMS Subject Classification 2000: 05C15, 05C40

1 Introduction

All graphs in this paper are finite, undirected, simple and connected. We follow the notation and terminology of [1]. Let c be a coloring of the edges of a graph G , i.e., $c : E(G) \longrightarrow \{1, 2, \dots, k\}$, $k \in \mathbb{N}$. A path is called a rainbow path if no two edges of the path have the same color. The graph G is called rainbow connected (with respect to c) if for every two vertices of G , there exists a rainbow path connecting them in G . If by coloring c the graph G is rainbow connected, then the coloring c is called a rainbow coloring of G . If k colors are used in c , then c is a rainbow k -coloring of G . The minimum

*Supported by NSFC.

number k for which there exists a rainbow k -coloring of G , is called the rainbow connection number of G , denoted by $rc(G)$.

Let c is a rainbow coloring of a graph G . If for every pair u and v of distinct vertices of the graph G , the graph G contains a rainbow u - v geodesic (a shortest path in G between v and u), then G is called strongly rainbow connected. In this case, the coloring c is called a strong rainbow coloring of G . If k colors are used, then c is a strong rainbow k -coloring of G . The minimum number k satisfying that G is strongly rainbow connected, i.e., the minimum number k for which there exists a strong rainbow k -coloring of G , is called the strong rainbow connection number of G , denoted by $src(G)$. Thus for every connected graph G , $rc(G) \leq src(G)$. Recall that the diameter of G is defined as the largest distance between two vertices of G , denoted $diam(G)$. Then $diam(G) \leq rc(G) \leq src(G)$. The following results were obtained in [2] by Chartrand et al.

Proposition 1.1 *Let G be a nontrivial connected graph of size m . Then*

1. $rc(G) = 1$ if and only if $src(G) = 1$.
2. $rc(G) = 2$ if and only if $src(G) = 2$.
3. $diam(G) \leq rc(G) \leq src(G)$ for every connected graph G . ■

Chartrand et al. also considered the problem that, given any two integers a and b , whether there exists a connected graph G such that $rc(G) = a$ and $src(G) = b$? and they got the following result.

Theorem 1.2 *Let a and b be positive integers with $a \geq 4$ and $b \geq (5a-6)/3$. Then there exists a connected graph G such that $rc(G) = a$ and $src(G) = b$. ■*

Then, combining Proposition 1.1 and Theorem 1.2, they got the following result.

Corollary 1.3 *Let a and b be positive integers. If $a = b$ or $3 \leq a < b$ and $b \leq \frac{5a-6}{3}$, then there exists a connected graph G such that $rc(G) = a$ and $src(G) = b$. ■*

Finally, they thought the question that whether the condition $b \leq \frac{5a-6}{3}$ can be deleted? and raised the following conjecture:

Conjecture 1.4 *Let a and b be positive integers. Then there exists a connected graph G such that $rc(G) = a$ and $src(G) = b$ if and only if $a = b \in \{1, 2\}$ or $3 \leq a \leq b$. ■*

This short note is to give a confirmative solution to this conjecture.

2 Proof of the conjecture

Proof of Conjecture 1.4: From Proposition 1.1 one can see that the condition is necessary. For the sufficiency, when $a = b \in \{1, 2\}$, from Corollary 1.3 the conjecture is true. So, we just need to consider the situation $3 \leq a \leq b$.

Let $n = 3b(b-a+2)$, and let H_n be the graph consisting of an n -cycle $C_n : v_1, v_2, \dots, v_n$ and another two vertices w and v , each of which joins to every vertex of C_n . Let G be the graph constructed from H_n of order $n + 2$ and the path $P_{a-1} : u_1, u_2, \dots, u_{a-1}$ on $a - 1$ vertices by identifying v and u_{a-1} .

First, we will show $rc(G) = a$. Because $\text{diam}(G) = a$, by Proposition 1.1 we have $rc(G) \geq a$. It remains to show $rc(G) \leq a$. Note that $n = 3b(b - a + 2) \geq 18$. Define a coloring c for the graph G by the following rules:

$$c(e) = \begin{cases} i & \text{if } e = u_i u_{i+1} \text{ for } 1 \leq i \leq a - 2, \\ a - 1 & \text{if } e = v_i v \text{ and } i \text{ is odd,} \\ a & \text{if } e = v_i v \text{ and } i \text{ is even,} \\ a & \text{if } e = v_i w \text{ and } 1 \leq i \leq n \\ 1 & \text{otherwise.} \end{cases}$$

Since c is a rainbow a -coloring of the edges of G , it follows that $rc(G) \leq a$. This implies $rc(G) = a$.

Next, we will show $src(G) = b$. We first show $src(G) \leq b$, by giving a strong rainbow b -coloring c for the graph G as follows:

$$c(e) = \begin{cases} i & \text{if } e = u_i u_{i+1} \text{ for } 1 \leq i \leq a - 2, \\ a - 2 + i & \text{if } e = v_{3b(i-1)+j} v \text{ for } 1 \leq i \leq b - a + 2 \text{ and } 1 \leq j \leq 3b, \\ i & \text{if } e = v_{3(j-1)b+3(i-1)+k} w \text{ for } 1 \leq j \leq b - a + 2 \text{ and } 1 \leq i \leq b \\ & \text{and } 1 \leq k \leq 3, \\ 1 & \text{if } e = v_{3(i-1)+1} v_{3(i-1)+2} \text{ for } 1 \leq i \leq b(b - a + 2), \\ 2 & \text{if } e = v_{3(i-1)+2} v_{3(i-1)+3} \text{ for } 1 \leq i \leq b(b - a + 2), \\ 3 & \text{otherwise} \end{cases}$$

It remains to show $src(G) \geq b$. By contradiction, suppose $rc(G) < b$. Then there exists a strong rainbow $(b - 1)$ -coloring $c : E(G) \rightarrow \{1, 2, \dots, b - 1\}$. For every v_i ($1 \leq i \leq n$), $d(v_i, u_1) = a - 1$, and the path $v_i v u_{a-2} \dots u_1$ is the only path of length $a - 1$ connecting v_i and u_1 , and so $v_i v u_{a-2} \dots u_1$ is a rainbow path. Without loss of generality, suppose $c(u_2 u_1) = 1$, $c(u_3 u_2) = 2, \dots, c(u_{a-1} u_{a-2}) = a - 2$. Then $c(v_i v) \in \{a - 1, a, \dots, b\}$, for $1 \leq i \leq n$. We first consider the set of edges $A = \{v_i v, 1 \leq i \leq n\}$, and so $|A| = n$. Thus there exist at least $\lceil \frac{n}{b-a+1} \rceil \geq 3b + 1$ edges in A colored the same. Suppose there exist m edges $v_{j_1} v, \dots, v_{j_m} v$, ($1 \leq j_1 < j_2 < \dots < j_m \leq n$) colored the same and

$m \geq \lceil \frac{n}{b-a+1} \rceil \geq 3b+1$. Second, we consider the set of edges $B = \{v_{j_1}w, \dots, v_{j_m}w\}$. Since $c(v_{j_i}w) \in \{1, 2, \dots, b-1\}$, for $1 \leq i \leq m$, then there exist at least $\lceil \frac{m}{b-1} \rceil \geq \lceil \frac{3b+1}{b-1} \rceil \geq 4$ edges colored the same. Thus from B we can choose 4 edges of the same color. Since $n \geq 18$, from the corresponding vertices on the cycle C_n of the four edges chosen above, we can get two vertices such that their distance on the cycle C_n is more than 3. Without loss of generality, we assume that the two vertices are v'_1, v'_2 and their distance in graph G is 2. Then the geodesic between v'_1 and v'_2 in graph G is either $v'_1wv'_2$ or $v'_1vv'_2$. However, neither $v'_1wv'_2$ nor $v'_1vv'_2$ is a rainbow path. Thus the coloring c is not a strong rainbow coloring of G , a contradiction. Therefore $src(G) \leq b$ and so $src(G) = b$. The proof is thus complete. ■

References

- [1] J.A. Bondy, U.S.R. Murty. *Graph Theory*, Springer, Heidelberg, 2008.
- [2] G. Chartrand, G.L. Johns, K.A. MeKeon, P. Zhang. Rainbow connection in graphs. *Math.Bohem.*, 133(1)(2008) 85-98.